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LETTER TO THE EDITOR

On the new algebra related to the non-standard *R*-matrix

S Meljanac† and M Milekovic‡

† Rudjer Boskovic Institute, Bijenicka c.54, 41001 Zagreb, Croatia
‡ Prirodoslovno-matematicki fakultet Zavod za teorijsku fiziku, Bijenicka c.32, 41000 Zagreb, Croatia

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Abstract. We demonstrate that the new algebra associated to the non-standard *R*-matrix, investigated previously by Fei and Yue is related to the $SU(2)_Q$-algebra by a simple transformation. Criteria which identify equivalent quantum algebras are given.

In a recent letter [1] Fei and Yue investigated a quantum enveloping algebra related to the $\hat{R}$-matrix (a solution of the braid group)

$$\hat{R}=\begin{pmatrix}1&&&\\&1+q&q&\\&-1&0&\\&&&1\end{pmatrix}. \tag{1}$$

They claim that the Fadeev–Reshetikhin–Takhtajan (FRT) quantization procedure [2], with (1) as input, yields a new quantum algebra. In this letter we explicitly show that the algebra found in [1] (equations (10)–(13)) is related to the well-known $SU(2)_Q$ algebra. We also give a simple criteria which allows one to identify isomorphic algebras.

We recall FRT-equations which connect quantum enveloping algebras and $\hat{R}$-matrices [2]

$$\begin{gathered}\hat{R}L_2(\varepsilon)L_1(\varepsilon')=L_2(\varepsilon')L_1(\varepsilon)\hat{R}\\ L_1(\varepsilon)=L(\varepsilon)\otimes 1 \qquad L_2(\varepsilon)=1\otimes L(\varepsilon)=PL_1(\varepsilon)P\\ (\varepsilon,\ \varepsilon')=(+,+),(-,-),(+,-)\\ L(+)=\begin{pmatrix}L^+_{11}&\cdots&L^+_{1n}\\0&&L^+_{nn}\end{pmatrix} \qquad L(-)=\begin{pmatrix}L^-_{11}&&0\\L^-_{n1}&\cdots&L^-_{nn}\end{pmatrix}\end{gathered} \tag{2}$$

where P is the permutation operator in $V\otimes V$.

The entries $L^{\pm}_{ij}$ of the upper ($L(+)$) and lower ($L(-)$) triangular matrices are generators of the quantum algebra $A\equiv A(\hat{R}, L^{\pm}_{ij})$ associated to a given $\hat{R}$-matrix.

The algebra $A(\hat{R}, L^{\pm}_{ij})$ may be endowed with a Hopf (co-algebraic) structure in the usual way, by defining the coproduct Δ, the co-unit ε and the antipode S [2].

Let $A\equiv A(\hat{R}, L^{\pm}_{ij})$ and $\mathscr{A}\equiv\mathscr{A}(\hat{\mathscr{R}}, \mathscr{L}^{\pm}_{ij})$ be two quantum algebras, obtainable from the FRT-procedure (2). Then the following observations hold (see also [3]):

(i) A and $\mathscr{A}$ are isomorphic as Hopf algebras if $\hat{R}$-matrices associated to them are related by the similarity (gauge) transformation of diagonal form (including scaling):

$$\hat{R}=\lambda W^{-1}\hat{\mathscr{R}}W \qquad \lambda\in C-\{0\}$$

(ii) The generators $L_{ij}^{\pm}$ (of A) and $\mathscr{L}_{ij}^{\pm}$ (of $\mathscr{A}$) are related as

$$L_1(\varepsilon)=PW^{-1}P\mathscr{L}_1(\varepsilon)W$$

$$L_2(\varepsilon)=W^{-1}\mathscr{L}_2(\varepsilon)PWP$$

(iii) The coproducts Δ (of A) and δ (of $\mathscr{A}$) are related as

$$\Delta=W^{-1}\delta W$$

Let us choose $\mathscr{A}\equiv\mathscr{A}(\hat{\mathscr{R}},\mathscr{L}_{ij}^{\pm})$ to be a one-parameter $SU(2)_Q$-algebra associated to the $\hat{\mathscr{R}}_Q$-matrix

$$\hat{\mathscr{R}}=\begin{pmatrix}1&&&\\&1-1/Q^2&1/Q&\\&1/Q&0&\\&&&1\end{pmatrix} \tag{3}$$

and let $A\equiv A(\hat{R},L_{ij}^{\pm}))$ be a two-parameter algebra $SU(2)_{p,k}$ [4] associated to the $\hat{R}_{p,k}$-matrix

$$\hat{R}=\begin{pmatrix}1&&&\\&1-1/pk&1/p&\\&1/k&0&\\&&&1\end{pmatrix}. \tag{4}$$

In an earlier paper [5] we showed that $SU(2)_{p,k}$ is isomorphic (as a Hopf algebra) to $SU(2)_Q$, with $Q^2=pk$. The similarity transformation which establishes this isomorphism is

$$\begin{aligned}&\hat{R}_{p,k}=W^{-1}(\eta)\hat{\mathscr{R}}_QW(\eta)\\&W(\eta)=\eta^{J_0/2}\otimes\eta^{-J_0/2}\\&J_0=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\\&\eta^2=p/k.\end{aligned} \tag{5}$$

With $k=-1, p=1/q$ we obtain the $\hat{R}$-matrix (1) from the $\hat{R}$-matrix (4). Owing to observations (i)–(iii), the algebras associated to the $\hat{R}$-matrices (1) or (4) are isomorphic to $SU(2)_Q$, with $Q^2=-1/q$. From (5) and observation (iii) one can easily derive relations between co-algebraic structures.

Finally, we point out that the Hopf algebra in [1] (equations (12) and (13)) directly follows from the two-parameter algebra $SU(2)_{p',k'}$, discussed in [5, 6], with $p'=-q^{1/2}$, $k'=q^{1/2}$, $Q'^2=p'k'=-q$, $\eta^2=p'/k'=-1$ and

$$\begin{aligned}&H=J_0\\&E^{\pm}=((q+1)/(q-1))^{1/2}(-1)^{-1/2(J_0\mp1/2)}J_{\mp}\end{aligned} \tag{6}$$

where J_0 and $J_\pm$ are generators of $SU(2)_{Q'}$. Hence, the algebra in [1], (equation (12)), is isomorphic to $SU(2)_{Q'}$ with $Q^{2'} = -q$. Then the co-algebra in [1] (equation (13)) is defined as in [5], i.e. $\Delta = \Delta_{Q'}$, $S = S_{Q'}$ and $\varepsilon = \varepsilon_{Q'}$, where the index Q' refers to $SU(2)_{Q'}$.

Notice that for $q = -1$ ($Q = \pm 1$) the algebra in [1], equations (12) and (13), is equivalent to $SU(2)_{Q=1}$ and $SU(2)_{Q=-1}$. It is interesting to observe that there is a general isomorphism between $SU(2)_q$ and $SU(2)_{-q}$, which can be established using the transformations as in [5] (see also [7]).

To conclude, we have found no evidence for the new algebraic (and co-algebraic) structure related to the $\hat{R}$-matrix (1). Nevertheless, it is still an interesting question whether non-trivial new algebras can be found in connection with other 4×4 solutions of the braid (Yang–Baxter) equations [8]. Our criteria (i)–(iii) significantly simplify such an analysis. This topic is currently under investigation.

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